

Virtual dipoles and large fluctuations in quantum gravity

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Abstract

The positive energy theorem precludes the possibility of Minkowski flat space decaying by any mechanism. In certain circumstances, however, large quantum fluctuations of the gravitational field could arise—not only at the Planck scale, but also at larger scales. This is because there exists a set of localised weak field configurations which satisfy the condition $\int d^4x \sqrt{g} R = 0$ and thus give a null contribution to the Einstein action. Such configurations can be constructed by solving Einstein field equations with unphysical dipolar sources. We discuss this mechanism and its modification in the presence of a cosmological term and/or an external field.

04.20.-q Classical general relativity.

04.60.-m Quantum gravity.

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1. Is Minkowski space stable?

Although flat spacetime is a trivial solution of the vacuum Einstein equations, the *stability* of this solution was questioned for a long time, mainly because in General Relativity (unlike in electrodynamics and other field theories) it is impossible to give a positive semidefinite expression for the field energy density. The numerous approaches to this problem reported in the literature can be essentially classified in the context of classical field theory, thermal field theory and quantum field theory. Let us quote very shortly some of the known facts.

(1) *Classical field theory* - The long-outstanding conjecture that the total energy (A.D.M. energy) of asymptotically flat manifolds is positive semidefinite and that only Minkowski space has zero energy was finally proven by Schoen and Yau in 1979 [1] and is now known as “the positive energy theorem”. Since energy is conserved, this theorem seems to preclude the possibility of flat space decaying by any mechanism.

(2) *Thermal field theory* - The instabilities of Minkowski space at a finite temperature were studied by Gross, Perry and Yaffe in 1982 [2]. They concluded that in hot flat space there are two distinct sources of instability: the large-wavelength density fluctuations of the thermal gravitons and the nucleation of black holes.

(3) *Quantum field theory* - One can consider, at least formally, the functional integral of the theory, which represents a sum over all possible field configurations weighed with the factor $\exp[i\hbar S] = \exp\left[\frac{-i\hbar}{16\pi G} \int d^4x \sqrt{g(x)} R(x)\right]$ and possibly with a factor due to the integration measure. The Minkowski space is a stationary point of the vacuum action and has maximum probability. “Off-shell” configurations, which are not solutions of the vacuum Einstein equations, are admitted in the functional integration but are strongly suppressed by the oscillations of $e^{i\hbar S}$.

In this letter (based in part upon our work [3]) we shall be concerned with the quantum case. Even though any tunnelling to a ground state different from Minkowski space appears to be impossible due to the positive energy theorem, we shall see that the quantum fluctuations about flat space can be enhanced in certain conditions.

2. The quantum of curvature fluctuation

Due to the appearance of the dimensional constant G in the Einstein action, the most probable quantum fluctuations of the gravitational field grow at very short distances, of the order of $L_{Planck} = \sqrt{G\hbar/c^3} \sim 10^{-33} \text{ cm}$. This led Hawking, Coleman and others to depict spacetime at

the Planck scale as a “quantum foam” [4], with high curvature and variable topology.

Let us reformulate this argument in short. Suppose we start with a flat configuration, then a curvature fluctuation appears in a region of size d . How much can the fluctuation grow before it is suppressed by the oscillating factor e^{iS} ? (We set $\hbar = 1$ and $c = 1$ in the following.) The contribution of the fluctuation to the action is of order Rd^4 ; both for positive and for negative R , the fluctuation is suppressed when this contribution exceeds ~ 1 in absolute value, therefore $|R|$ cannot exceed $\sim G/d^4$. This means that the fluctuations of R are stronger at short distances—down to L_{Planck} , the minimum physical distance.

Clearly if the curvature is large, then the metric is locally far from being flat and the factor \sqrt{g} in the action is not trivially ~ 1 , thus the estimate above is only approximate. In the following, however, we shall focus on the case of a weak field with small curvature, at distances much larger than L_{Planck} (without any topology change). In this case we could say that $R_0 \equiv G/d^4$ represents the “quantum of curvature fluctuation”, very small at macroscopic scale. As shown above, the number of such quanta in a certain region does not exceed $N \sim 1$; therefore fluctuations are practically irrelevant and spacetime looks almost perfectly flat at distances much larger than L_{Planck} .

3. Virtual dipoles containing N $+/$ quanta and N $-/$ quanta

At this point one might rise the following objection. Suppose two quanta of curvature fluctuation with opposite signs pop up in flat space, in two adjacent regions having the same size d . This does not change the total action. Then the negative fluctuation can grow up to comprise 2, 3, ... N quanta, provided the same happens with the positive fluctuation, because the total action of a configuration containing N $+/$ quanta and N $-/$ quanta is the same as the flat space action. Can this represent a possible instability, a way for the gravitational field to “run away” from the flat configuration, causing strong fluctuations of the metric also at scales larger than Planck scale?

This idea might seem naive and qualitative, and perhaps every beginner in General Relativity had it for a minute. However, it can be precised and made more rigorous. It is possible to construct explicitly field configurations with the property above, namely having scalar curvature which vanishes identically almost everywhere, except in two adjacent regions – one with positive R and the other with negative R – in such a way that the total integral of $\sqrt{g}R$ is zero.

One can regard each of these field configurations as the field generated by a virtual “mass

dipole". In fact, they can be defined as the solutions of the Einstein field equations with a dipolar source—a positive and a negative mass with certain sizes, chosen in such a way that the total integral of the scalar curvature is zero. Such sources are clearly unphysical and do not exist in the real world; however, here we are not interested into a solution of the classical field equations with physical sources but into any field configuration which can cause strong fluctuations in the functional integral.

Let us consider a solution $g_{\mu\nu}(x)$ of the Einstein equations

$$R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) = -8\pi GT_{\mu\nu}(x), \quad (1)$$

with a (covariantly conserved) source $T_{\mu\nu}(x)$ obeying the additional integral condition

$$\int d^4x \sqrt{g(x)} \text{Tr } T(x) = 0. \quad (2)$$

Taking into account the trace of eq. (1), namely $R(x) = 8\pi G \text{Tr } T(x)$, we see that the action $\int d^4x \sqrt{g} R$ computed for this solution is zero.

As an example of an unphysical source which satisfies (2) one can consider a static dipole centred at the origin ($m, m' > 0$):

$$T_{\mu\nu}(\mathbf{x}) = \delta_{\mu 0} \delta_{\nu 0} [m f(\mathbf{x} + \mathbf{a}) - m' f(\mathbf{x} - \mathbf{a})]. \quad (3)$$

Here $f(\mathbf{x})$ is a smooth test function centred at $\mathbf{x} = 0$, rapidly decreasing and normalised to 1, which represents the mass density. The range of f , say r_0 , is such that $a \gg r_0 \gg r_{Schw}$, where r_{Schw} is the Schwarzschild radius corresponding to the mass m . The mass m' is in general different from m and chosen in such a way to compensate the small difference, due to the \sqrt{g} factor, between the integrals

$$I^+ = \int d^3x \sqrt{g(\mathbf{x})} f(\mathbf{x} + \mathbf{a}) \quad \text{and} \quad I^- = \int d^3x \sqrt{g(\mathbf{x})} f(\mathbf{x} - \mathbf{a}). \quad (4)$$

The procedure for the construction of the desired field configuration is the following. One first considers Einstein equations with the source (3). Then one solves them with a suitable method, for instance in the weak field approximation. Finally, knowing $\sqrt{g(x)}$ one computes the two integrals (4) and adjusts the parameter m' in such a way that

$$mI^+ - m'I^- = 0 \quad (5)$$

(see [3], where these configurations were called “zero modes of the Einstein action”). To first order in G , the relation between m and m' turns out to be

$$m' \simeq m(1 + 8\pi Gm). \quad (6)$$

Note that these field configurations are in no way singular, so we do not expect them to be suppressed by the functional integration measure.

4. Does the effective gravitational lagrangian contain a scale-dependent cosmological term?

What can stabilise the Einstein action with respect to the dipole fluctuations described above? Possibly an additional term in the lagrangian. R^2 terms are only relevant at very short distances, however, which is not the case here. Another typical addition is the cosmological term $(\Lambda/8\pi G) \int d^4x \sqrt{g}$. It is immediate to check, using eq.s (4)–(6) above, that such a term receives non-vanishing contributions from dipole fluctuations, and will therefore suppress them.

It is known that Λ , if not zero, is very small in our universe; its effective value, however, could depend on the scale, being very small at cosmological distances but somewhat larger at short distances.

This concept, namely that the gravitational lagrangian may comprise a scale-dependent cosmological term, originally emerged in the Euclidean theory of gravity on the Regge lattice. Recent non-perturbative numerical simulations [5] allow to study a “discretized spacetime” whose dynamics is governed by an action containing G and Λ as bare parameters; it turns out that as the continuum limit is approached, the adimensional product $|\Lambda_{eff}|G_{eff}$ behaves like

$$|\Lambda_{eff}|G_{eff} \sim (l_0/l)^\gamma \quad (7)$$

where l is the scale, l_0 is the lattice spacing, γ a critical exponent and the sign of Λ_{eff} is negative (for an earlier discussion see [6]). Furthermore, one can reasonably assume that $l_0 \sim L_{Planck}$, and that the scale dependence of G_{eff} is much weaker than that of Λ_{eff} .

A scale dependence of Λ_{eff} like that in eq. (7) also implies that any bare value of Λ , expressing the energy density associated to the vacuum fluctuations of the quantum fields including the gravitational field itself, is “relaxed to zero” at long distances just by virtue of the gravitational dynamics, without any need of a fine tuning. One would have, in other words, a purely gravitational solution of the cosmological constant paradox.

We do not intend to discuss here whether (and for what values of γ) an *ansatz* like eq. (7), with $\Lambda_{eff} < 0$, is compatible with the most recent estimates of the Hubble parameter [7]. In fact, admitted that Λ_{eff} depends on the scale l , it is clear that the determination of this dependence and a comparison with the observational constraints is a complex problem, given

the enormous range spanned by l . One can consider at least four domains: a cosmological scale, a scale of the order of the solar system size (compare [8]), a laboratory scale (in a wide sense, i.e., down to the subnuclear and GeV scale [9]) and the Planck scale. While in this work we are mostly concerned with the laboratory scale, we shall keep a general approach and just denote by $\Lambda_{eff}(l)$ the general unknown function which gives the scale dependence of Λ_{eff} .

5. Coupling to an external field

We have seen that a scale-dependent cosmological term in the effective gravitational action is able to suppress the virtual dipole fluctuations. The actual existence of this term can be regarded as a consistent hypothesis, suggested by some results of lattice theory. But how can we check that the whole idea makes sense, i.e. that on one hand “dangerous” dipole fluctuations are admitted by the Einstein action and on the other hand an intrinsic cosmological term is there to suppress them?

We can imagine a situation in which this latter term is canceled, in some region of spacetime, by coupling gravity to a suitable *external* field $F(x, t)$. Then, if virtual dipoles really exist, they will be free to grow in this region and cause abnormally large fluctuations. The amplitude of these fluctuations cannot be predicted at this stage, but eventually they could lead to a sort of partial “thermalization” of the gravitational field in this region.

The function $F(x, t)$ represents an assigned classical field, not a variable of the functional integral. It couples to gravity through its energy-momentum tensor

$$T_{\mu\nu} = \Pi_\mu \partial_\nu F - g_{\mu\nu} \mathcal{L} \quad (8)$$

where Π_μ is the canonically conjugated momentum $\delta\mathcal{L}/\delta(\partial^\mu F)$. For instance, for a classical scalar field ϕ (typically describing, in the context of quantum field theory, coherent matter of some kind) we have

$$\mathcal{L} = \frac{1}{2} \left(\partial_\alpha \phi \partial^\alpha \phi - m^2 \phi^2 \right), \quad (9)$$

$$\Pi_\mu = \partial_\mu \phi, \quad (10)$$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L} \quad (11)$$

and the interaction term in the action takes the form

$$S' = 8\pi G \int d^4x h_{\mu\nu} T^{\mu\nu} = 8\pi G \int d^4x (h_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - \text{Tr} h \mathcal{L}). \quad (12)$$

The term $h_{\mu\nu}\partial^\mu\phi\partial^\nu\phi$ can be regarded as a source term for $h_{\mu\nu}$, while the term $\int d^4x \text{Tr} h \mathcal{L}$ interferes with the cosmological term: we recall that, to first order in h , $\sqrt{g} \simeq 1 + \frac{1}{2}\text{Tr} h$; therefore, if at some point x the condition

$$\frac{\Lambda_{eff}(a)}{8\pi G} + \mathcal{L}(\phi(x)) = 0 \quad (13)$$

is satisfied, at that point large virtual dipoles fluctuations of scale a can appear.

Eq. (13) can be regarded as a parametric equation for $a(x)$: given the value of the coherent lagrangian density at x , one finds a corresponding value for Λ_{eff} and thus for the scale a of the dipole fluctuations allowed at x . (This makes sense if ϕ varies on a scale much larger than a . Also note that being ϕ an assigned “off shell” field, $\mathcal{L}(\phi(x))$ is not necessarily an extremal value.) The scale obtained in this way can turn out to be physically relevant, or not—particularly if very short.

6. Conclusions

This work aims at pointing out a peculiar quantum behaviour of the gravitational field which is usually disregarded, but cannot a priori be excluded. The possibility of anomalous growth of the “dipole fluctuations” described above appears to be an intrinsic property of the Einstein action.

It is therefore important to understand to what extent these fluctuations are affected by a cosmological term in the action or by the coupling to an external field—which breaks the translation symmetry of Minkowski spacetime and could thus trigger the fluctuations in certain regions.

Our analysis is limited by the poor present knowledge of the quantum dynamics of the gravitational field; in particular, it seems impossible to predict at this stage the exact amplitude of the dipole fluctuations and the scale dependence $\Lambda_{eff}(l)$ (if any).

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